Problem Set: The Diamond Overlapping-Generations Model

Advanced Macroeconomics — Dr Lei Pan — Total: 100 Marks

Instructions. Answer all questions. Show all mathematical derivations clearly. Answers without derivation receive limited credit. Time is discrete, $t = 0, 1, 2, \ldots$. Individuals live for two periods. Population and technology satisfy

$$L_t = (1+n)L_{t-1}, \qquad A_{t+1} = (1+g)A_t, \qquad D \equiv (1+n)(1+g).$$

There is no capital depreciation. Define capital per unit of effective labour as

$$k_t \equiv \frac{K_t}{A_t L_t}.$$

Question 1: Competitive Equilibrium and Endogenous Saving [Total: 60 marks]

Consider the Diamond economy in which a young individual born at t has lifetime utility

$$U(c_{1t}, c_{2,t+1}) = u(c_{1t}) + \beta u(c_{2,t+1}), \qquad 0 < \beta < 1,$$

with CRRA period utility

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \qquad \theta > 0,$$

and $u(c) = \ln c$ when $\theta = 1$. The individual budget constraints are

$$c_{1t} + s_t = A_t w_t, \qquad c_{2,t+1} = (1 + r_{t+1})s_t.$$

Firms operate the CRS technology

$$Y_t = F(K_t, A_t L_t^D) = A_t L_t^D f(k_t), \qquad k_t = \frac{K_t}{A_t L_t^D}.$$

(a) Derive the Euler equation for the young individual's saving problem. Then derive the saving function

$$s_t = \sigma(r_{t+1})A_t w_t, \qquad \sigma(r) = \frac{\beta^{1/\theta}}{\beta^{1/\theta} + (1+r)^{1-1/\theta}}.$$

Explain briefly why the Inada condition rules out zero consumption.

(b) Derive $\sigma'(r)$ and determine whether the saving rate rises or falls with the interest rate when $\theta < 1$, $\theta = 1$, and $\theta > 1$. Give the economic interpretation in terms of substitution and income effects.

(c) Solve the firm's profit-maximisation problem. Derive

$$r_t = f'(k_t), \qquad w_t = f(k_t) - f'(k_t)k_t.$$

Then show why CRS and perfect competition imply zero profits.

(d) Use labour-market clearing $L_t^D = L_t$ and capital-market clearing $K_{t+1} = L_t s_t$ to derive the implicit transition equation

$$k_{t+1} = \frac{1}{D}\sigma\big(f'(k_{t+1})\big)\left[f(k_t) - f'(k_t)k_t\right].$$

Explain why this equation fully characterises the equilibrium path of k_t.

Question 2: Log Utility, Steady State, Golden Rule, and Dynamic Inefficiency [Total: 40 marks]

Assume now that $\theta = 1$ and

$$f(k) = k^\alpha, \qquad 0 < \alpha < 1.$$

Hence $\sigma(r) = \beta/(1+\beta)$.

(a) Derive the explicit transition equation

$$k_{t+1} = Bk_t^\alpha, \qquad B \equiv \frac{\beta}{1+\beta}\frac{1-\alpha}{D}.$$

Solve for the unique positive steady state k^* and prove global stability. Derive the log-linear convergence coefficient and the half-life of deviations from steady state.

(b) For

$$\alpha = \frac{1}{3}, \qquad \beta = 0.96, \qquad n = 0.01, \qquad g = 0.02,$$

compute D, B, k^*, y^*, w^*, and r^*. State the balanced-growth-path growth rates of aggregate output, aggregate capital, aggregate consumption, output per worker, and capital per worker.

(c) Derive the Golden Rule capital stock. Starting from the aggregate resource constraint with no depreciation, show that steady-state consumption per unit of effective labour is

$$c(k) = f(k) - (D-1)k.$$

Then derive

$$f'(k_{\text{GR}}) = D - 1, \qquad k_{\text{GR}} = \left(\frac{\alpha}{D-1}\right)^{1/(1-\alpha)}.$$

State the dynamic-efficiency condition.

(d) Using the numerical parameters in part (b), compare k^* with k_{GR} and compare r^* with $D-1$. Is the economy dynamically inefficient? Then derive the general condition for overaccumulation in the log–Cobb–Douglas Diamond economy.

Detailed Solutions

Solution to Question 1 [60 marks]

Part (a) [18 marks]

The young individual chooses saving s_t to solve

$$\max_{s_t} u(A_t w_t - s_t) + \beta u\big((1+r_{t+1})s_t\big).$$

The two consumptions are

$$c_{1t} = A_t w_t - s_t, \qquad c_{2,t+1} = (1+r_{t+1})s_t.$$

The first-order condition is

$$-u'(c_{1t}) + \beta u'(c_{2,t+1})(1+r_{t+1}) = 0.$$

Hence,

$$\boxed{u'(c_{1t}) = \beta(1+r_{t+1})u'(c_{2,t+1}).}$$

Equivalently,

$$\boxed{\frac{u'(c_{1t})}{\beta u'(c_{2,t+1})} = 1 + r_{t+1}.}$$

This is the consumption Euler equation. It states that the marginal utility cost of reducing young-age consumption by one unit must equal the discounted marginal utility benefit of increasing old-age consumption by the gross return $1 + r_{t+1}$.
For CRRA utility,

$$u'(c) = c^{-\theta}.$$

Therefore, the Euler equation becomes

$$c_{1t}^{-\theta} = \beta(1+r_{t+1})c_{2,t+1}^{-\theta}.$$

Using

$$c_{1t} = A_t w_t - s_t, \qquad c_{2,t+1} = (1+r_{t+1})s_t,$$

we obtain

$$(A_t w_t - s_t)^{-\theta} = \beta(1+r_{t+1})\left[(1+r_{t+1})s_t\right]^{-\theta}.$$

Thus,

$$(A_t w_t - s_t)^{-\theta} = \beta(1+r_{t+1})^{1-\theta} s_t^{-\theta}.$$

Taking both sides to the power $-1/\theta$ gives

$$A_t w_t - s_t = \beta^{-1/\theta}(1+r_{t+1})^{1-1/\theta} s_t.$$

Hence,

$$A_t w_t = s_t\left[1 + \beta^{-1/\theta}(1+r_{t+1})^{1-1/\theta}\right].$$

Therefore,

$$s_t = \frac{A_t w_t}{1 + \beta^{-1/\theta}(1+r_{t+1})^{1-1/\theta}}.$$

Multiplying numerator and denominator by $\beta^{1/\theta}$,

$$\boxed{s_t = \frac{\beta^{1/\theta}}{\beta^{1/\theta} + (1+r_{t+1})^{1-1/\theta}} A_t w_t.}$$

Thus,

$$\boxed{s_t = \sigma(r_{t+1})A_t w_t, \qquad \sigma(r) = \frac{\beta^{1/\theta}}{\beta^{1/\theta} + (1+r)^{1-1/\theta}}.}$$

The Inada condition,

$$\lim_{c\to 0} u'(c) = +\infty,$$

rules out zero consumption because if $c_{1t} = 0$, then the marginal utility of young-age consumption is infinite. The Euler equation cannot hold at such a corner when income is positive. Similarly, $c_{2,t+1} = 0$ cannot be optimal because the marginal utility of old-age consumption would be infinite. Hence the optimum is interior:

$$c_{1t} > 0, \qquad c_{2,t+1} > 0.$$

Marking guide: setup of saving problem, 3; Euler equation, 5; CRRA saving function, 7; Inada/interior-solution explanation, 3.

Part (b) **[10 marks]**

Let

$$B_\beta \equiv \beta^{1/\theta}, \qquad a \equiv 1 - \frac{1}{\theta}.$$

Then

$$\sigma(r) = \frac{B_\beta}{B_\beta + (1+r)^a}.$$

Differentiate:

$$\sigma'(r) = -\frac{B_\beta a (1+r)^{a-1}}{[B_\beta + (1+r)^a]^2}.$$

Since

$$a = 1 - \frac{1}{\theta}, \qquad -a = \frac{1}{\theta} - 1, \qquad a - 1 = -\frac{1}{\theta},$$

we obtain

$$\boxed{\sigma'(r) = \left(\frac{1}{\theta} - 1\right) \frac{\beta^{1/\theta}(1+r)^{-1/\theta}}{\left[\beta^{1/\theta} + (1+r)^{1-1/\theta}\right]^2}.}$$

Because the fraction after the first bracket is strictly positive,

$$\operatorname{sign} \sigma'(r) = \operatorname{sign}\left(\frac{1}{\theta} - 1\right).$$

Therefore,

$$\boxed{\sigma'(r) \begin{cases} > 0, & \theta < 1, \\ = 0, & \theta = 1, \\ < 0, & \theta > 1. \end{cases}}$$

If $\theta < 1$, the intertemporal elasticity of substitution is $1/\theta > 1$. The substitution effect dominates: a higher interest rate makes future consumption cheaper relative to current consumption, so the young save more.
If $\theta > 1$, the intertemporal elasticity of substitution is $1/\theta < 1$. The income effect dominates for a net saver: a higher interest rate makes it possible to finance old-age consumption with less saving, so the saving rate falls.
If $\theta = 1$, the two effects exactly cancel:

$$\boxed{\sigma(r) = \frac{\beta}{1+\beta}.}$$

Marking guide: correct derivative, 5; correct sign cases, 3; economic interpretation, 2.

Part (c) **[12 marks]**

The representative competitive firm solves

$$\max_{K_t, L_t^D} F(K_t, A_t L_t^D) - r_t K_t - w_t A_t L_t^D.$$

Using CRS, write output as

$$F(K_t, A_t L_t^D) = A_t L_t^D f(k_t), \qquad k_t = \frac{K_t}{A_t L_t^D}.$$

Thus profits are

$$\pi_t = A_t L_t^D \left[f(k_t) - r_t k_t - w_t\right].$$

The first-order condition with respect to K_t gives

$$\boxed{r_t = f'(k_t).}$$

The first-order condition with respect to effective labour gives

$$\boxed{w_t = f(k_t) - f'(k_t) k_t.}$$

To show zero profits, substitute the FOCs into profits:

$$\pi_t = A_t L_t^D \left[f(k_t) - f'(k_t)k_t - (f(k_t) - f'(k_t)k_t) \right].$$

Therefore,

$$\boxed{\pi_t = 0.}$$

Economically, CRS plus perfect competition implies that factor payments exhaust output. Capital is paid its marginal product and effective labour receives the residual marginal product, leaving no pure profits.
Marking guide: firm problem, 3; rental-rate FOC, 3; wage FOC, 3; zero-profit result and interpretation, 3.

Part (d) [20 marks]

Labour-market clearing requires

$$L_t^D = L_t.$$

Capital-market clearing requires next period's capital stock to equal total saving by the young:

$$K_{t+1} = L_t s_t.$$

Using the individual saving function,

$$s_t = \sigma(r_{t+1}) A_t w_t,$$

we get

$$K_{t+1} = L_t \sigma(r_{t+1}) A_t w_t.$$

Divide both sides by $A_{t+1}L_{t+1}$:

$$\frac{K_{t+1}}{A_{t+1}L_{t+1}} = \frac{L_t A_t}{A_{t+1}L_{t+1}} \sigma(r_{t+1}) w_t.$$

Since

$$A_{t+1}L_{t+1} = (1+g)(1+n)A_t L_t = D A_t L_t,$$

we have

$$\boxed{k_{t+1} = \frac{1}{D}\sigma(r_{t+1}) w_t.}$$

From firm optimality,

$$r_{t+1} = f'(k_{t+1}), \qquad w_t = f(k_t) - f'(k_t)k_t.$$

Substituting these expressions gives

$$\boxed{k_{t+1} = \frac{1}{D}\sigma(f'(k_{t+1})) \left[f(k_t) - f'(k_t)k_t \right].}$$

For CRRA preferences,

$$\sigma(f'(k_{t+1})) = \frac{\beta^{1/\theta}}{\beta^{1/\theta} + \left[1 + f'(k_{t+1})\right]^{1-1/\theta}}.$$

Therefore,

$$\boxed{k_{t+1} = \frac{1}{D} \frac{\beta^{1/\theta}}{\beta^{1/\theta} + \left[1 + f'(k_{t+1})\right]^{1-1/\theta}} \left[f(k_t) - f'(k_t)k_t \right].}$$

This equation fully characterises equilibrium dynamics because, given k_0 and the exogenous paths of A_t and L_t, it determines $k_1, k_2, \ldots$. Once $\{k_t\}$ is known, firm FOCs determine $\{r_t, w_t\}$, household optimisation determines $\{s_t, c_{1t}, c_{2,t+1}\}$, and aggregate variables follow from production and market clearing.
Marking guide: labour and capital clearing, 4; division by $A_{t+1}L_{t+1}$, 4; substitution of firm FOCs, 5; CRRA transition equation, 4; interpretation of equilibrium dynamics, 3.

Solution to Question 2 [40 marks]

Part (a) [16 marks]

With log utility, $\theta = 1$, the saving rate is constant:

$$\boxed{\sigma(r) = \frac{\beta}{1+\beta}.}$$

With Cobb–Douglas production,

$$f(k) = k^\alpha.$$

Then

$$f'(k) = \alpha k^{\alpha-1}.$$

The wage per unit of effective labour is

$$w_t = f(k_t) - f'(k_t)k_t.$$

Substitute:

$$w_t = k_t^\alpha - \alpha k_t^{\alpha-1}k_t = (1-\alpha)k_t^\alpha.$$

The transition equation becomes

$$k_{t+1} = \frac{1}{D}\frac{\beta}{1+\beta}(1-\alpha)k_t^\alpha.$$

Define

$$\boxed{B \equiv \frac{\beta}{1+\beta}\frac{1-\alpha}{D}.}$$

Therefore,

$$\boxed{k_{t+1} = Bk_t^\alpha.}$$

A positive steady state satisfies

$$k^* = B(k^*)^\alpha.$$

For $k^* > 0$,

$$(k^*)^{1-\alpha} = B.$$

Thus,

$$\boxed{k^* = B^{1/(1-\alpha)}.}$$

To prove global stability, take logs:

$$\ln k_{t+1} = \ln B + \alpha \ln k_t.$$

At the steady state,

$$\ln k^* = \ln B + \alpha \ln k^*.$$

Subtract the second equation from the first:

$$\ln k_{t+1} - \ln k^* = \alpha(\ln k_t - \ln k^*).$$

Define

$$\widehat{k}_t \equiv \ln k_t - \ln k^*.$$

Then

$$\boxed{\widehat{k}_{t+1} = \alpha \widehat{k}_t.}$$

Because $0 < \alpha < 1$, deviations shrink monotonically in logs:

$$\widehat{k}_t = \alpha^t \widehat{k}_0 \to 0.$$

Hence the steady state is globally stable for any $k_0 > 0$.
The log-linear convergence coefficient is

$$\boxed{\lambda = \alpha.}$$

The half-life h solves

$$\alpha^h = \frac{1}{2}.$$

Therefore,

$$\boxed{h = \frac{\ln(1/2)}{\ln \alpha}.}$$

Marking guide: log-utility saving rate, 2; explicit transition equation, 5; steady state, 3; global-stability proof, 4; convergence coefficient and half-life, 2.

Part (b) **[10 marks]**

Given

$$\alpha = \frac{1}{3}, \qquad \beta = 0.96, \qquad n = 0.01, \qquad g = 0.02,$$

we have

$$D = (1+n)(1+g) = (1.01)(1.02) = 1.0302.$$

Also,

$$B = \frac{\beta}{1+\beta}\frac{1-\alpha}{D} = \frac{0.96}{1.96}\frac{2/3}{1.0302} \approx 0.3170.$$

The steady state is

$$k^* = B^{1/(1-\alpha)} = B^{3/2}.$$

Thus,

$$\boxed{k^* \approx 0.1784.}$$

Output per unit of effective labour is

$$y^* = (k^*)^\alpha = (k^*)^{1/3},$$

so

$$\boxed{y^* \approx 0.5630.}$$

The wage per unit of effective labour is

$$w^* = (1-\alpha)(k^*)^\alpha = \frac{2}{3}y^*,$$

so

$$\boxed{w^* \approx 0.3753.}$$

The interest rate is

$$r^* = f'(k^*) = \alpha(k^*)^{\alpha-1}.$$

Hence,

$$\boxed{r^* \approx 1.0517.}$$

The gross return is

$$1 + r^* \approx 2.0517.$$

On the balanced growth path, $k_t = k^*$ is constant. Hence,

$$K_t = A_t L_t k^*, \qquad Y_t = A_t L_t y^*.$$

Therefore aggregate capital and aggregate output grow at the exact gross rate

$$D = (1+n)(1+g),$$

or exact net rate

$$D - 1 = n + g + ng.$$

Aggregate consumption and investment also grow at rate $D - 1$. Output per worker and capital per worker grow at rate g because

$$\frac{Y_t}{L_t} = A_t y^*, \qquad \frac{K_t}{L_t} = A_t k^*.$$

Numerically,

$$D - 1 = 0.0302,$$

so aggregate variables grow at 3.02%, while per-worker variables grow at 2%.
Marking guide: D and B, 2; k^* and y^*, 3; w^* and r^*, 3; BGP growth rates, 2.

Part (c) **[9 marks]**

Because there is no depreciation, the aggregate resource constraint is

$$C_t + K_{t+1} = F(K_t, A_t L_t) + K_t.$$

Divide by $A_t L_t$:

$$\frac{C_t}{A_t L_t} + \frac{K_{t+1}}{A_t L_t} = f(k_t) + k_t.$$

Since

$$\frac{K_{t+1}}{A_tL_t} = \frac{K_{t+1}}{A_{t+1}L_{t+1}} \frac{A_{t+1}L_{t+1}}{A_tL_t} = Dk_{t+1},$$

we obtain

$$c_t = f(k_t) + k_t - Dk_{t+1},$$

where

$$c_t \equiv \frac{C_t}{A_tL_t}.$$

In steady state,

$$k_{t+1} = k_t = k.$$

Therefore,

$$\boxed{c(k) = f(k) - (D-1)k.}$$

The Golden Rule capital stock maximises $c(k)$:

$$\max_k \left[f(k) - (D-1)k\right].$$

The first-order condition is

$$f'(k_{\text{GR}}) - (D-1) = 0.$$

Hence,

$$\boxed{f'(k_{\text{GR}}) = D - 1.}$$

For Cobb–Douglas production,

$$f'(k) = \alpha k^{\alpha-1}.$$

Thus,

$$\alpha k_{\text{GR}}^{\alpha-1} = D - 1.$$

Equivalently,

$$k_{\text{GR}}^{1-\alpha} = \frac{\alpha}{D-1}.$$

Therefore,

$$\boxed{k_{\text{GR}} = \left(\frac{\alpha}{D-1}\right)^{1/(1-\alpha)}.}$$

The dynamic-efficiency condition can be written as

$$\boxed{k^* \leq k_{\text{GR}}.}$$

Equivalently,

$$\boxed{f'(k^*) \geq D - 1.}$$

Since $r^* = f'(k^*)$, this is

$$\boxed{r^* \geq D - 1.}$$

If

$$k^* > k_{\text{GR}},$$

or equivalently

$$r^* < D - 1,$$

the economy has overaccumulated capital and is dynamically inefficient.
Marking guide: resource constraint, 3; steady-state consumption function, 2; Golden Rule FOC, 2; Cobb–Douglas k_{GR} and efficiency condition, 2.

Part (d) **[5 marks]**

Using the numerical parameters,

$$D - 1 = 1.0302 - 1 = 0.0302.$$

The Golden Rule capital stock is

$$k_{\text{GR}} = \left(\frac{1/3}{0.0302}\right)^{3/2} \approx 36.6697.$$

From part (b),

$$k^* \approx 0.1784.$$

Therefore,

$$\boxed{k^* < k_{\mathrm{GR}}.}$$

Also,

$$r^* \approx 1.0517$$

and

$$D - 1 = 0.0302.$$

Hence,

$$\boxed{r^* > D - 1.}$$

The economy is not dynamically inefficient. It has capital underaccumulation relative to the Golden Rule.
Now derive the general condition for overaccumulation. In the log–Cobb–Douglas economy,

$$k^* = \left[\frac{\beta}{1+\beta}\frac{1-\alpha}{D}\right]^{1/(1-\alpha)}.$$

Let

$$q \equiv \frac{\beta}{1+\beta}.$$

Then

$$k^* = \left[\frac{q(1-\alpha)}{D}\right]^{1/(1-\alpha)}.$$

The Golden Rule capital stock is

$$k_{\mathrm{GR}} = \left[\frac{\alpha}{D-1}\right]^{1/(1-\alpha)}.$$

Since $1/(1-\alpha) > 0$,

$$k^* > k_{\mathrm{GR}}$$

if and only if

$$\frac{q(1-\alpha)}{D} > \frac{\alpha}{D-1}.$$

Thus, dynamic inefficiency occurs if and only if

$$\boxed{q(1-\alpha)(D-1) > \alpha D, \qquad q = \frac{\beta}{1+\beta}.}$$

Equivalently,

$$\boxed{\frac{\beta}{1+\beta}(1-\alpha)(D-1) > \alpha D.}$$

For the numerical parameters, this inequality is strongly violated, so the economy is dynamically efficient.
Marking guide: numerical k_{GR}, 1; comparison of k^* and k_{GR}, 1; comparison of r^* and $D-1$, 1; general overaccumulation condition, 2.